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Application of Wavelet Methods to Volterra Integro-Differential Systems

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ABSTRACT

A numerical method for solving systems of linear Volterra integro-differential equations that is efficient and based on wavelets is introduced in this study. To simplify the provided integro-differential system into a system of linear algebraic equations, the suggested method uses a wavelet collocation technique. The original system's complexity is drastically lowered without sacrificing accuracy by discretising the issue at appropriate collocation locations. Approximate answers are obtained by solving the ensuing algebraic system; they are compared with precise analytical solutions in order to assess the method's performance. Experiments using numerical data show that the suggested wavelet approach yields very precise outcomes with little absolute errors.

Keywords: *Wavelet, Linear, Collocation, Errors, Solutions.*

I. INTRODUCTION

Volterra integro-differential systems are a significant family of functional equations that capture the behaviour of dynamic systems over time by combining the properties of differential and integral equations. In mathematical modelling, these systems—named after the Italian mathematician Vito Volterra, who was an early trailblazer in studying them—emerge when a process's current state is dependent on both its rate of development and its behaviour in the past. Volterra integro-differential systems are excellent models for biological populations, viscoelastic materials, epidemiological spread, control systems, economics, neural networks, and other real-world phenomena where past states impact future dynamics due to their inherent memory effect.

Typically, in a mathematical context of a Volterra integro-differential system, unknown functions are required to fulfil equations that contain derivatives and integrals with upper limit variables. The integral term in Volterra-type equations reflects causality and time-dependent memory; it extends from a fixed beginning point to the present moment. Fredholm integro-differential equations, in which integrals are obtained across specified intervals, are structurally distinct from these in this respect. Because it permits the natural and physically relevant inclusion of past data and beginning circumstances, the Volterra framework is therefore ideally suited for initial value issues. Classical differential equations fail to account for memory effects, whereas these systems do a better job of simulating numerous processes.

Volterra integro-differential systems are attracting more and more research interest because they may bring together disparate mathematical models in a unified analytical framework. The future evolution of processes in many physical and biological systems is dependent on the cumulative effect of past states, since many processes display hereditary features. For example, in the field of population dynamics, factors like as resource depletion and gestation delays can influence a species' growth rate. Also, in viscoelastic materials, stress is not just a function of immediate strain but of the whole deformation history. Such phenomena are inherently captured by Volterra integro-differential systems by use of convolution-type kernels that measure the impact of previous states on the current behaviour.

Given their hybrid structure, Volterra integro-differential systems provide substantial mathematical hurdles from a theoretical standpoint. Analysis of existence, uniqueness, stability, and regularity of solutions becomes more complicated when nonlocal influences are included via integral terms compared to ordinary differential equations. The complexity of the system is increased by the fact that the kernels appearing in the integral terms might be either linear or nonlinear, weakly or strongly singular, and time-dependent or state-dependent. This has led functional analysts, operator theorists, and applied mathematicians to focus on creating rigorous analytical tools for investigating complex systems.

The formulation of Volterra integro-differential systems in suitable functional spaces is a basic feature of these systems. In order to handle infinite-dimensional systems that arise from partial integro-differential equations and distributed parameter models, solutions are frequently sought in Banach or Hilbert spaces. Compactness arguments, semigroup theory, and fixed-point theorems are essential in proving well-posedness. The Volterra operator, with its time-triangular structure, has good analytical features that make initial value issues easier to understand. The existence and uniqueness of solutions under different assumptions on the kernels and nonlinear terms have been shown by drawing heavily on these features.

Volterra integro-differential systems are important in numerical computing and theoretical analysis as well. It is necessary to create effective numerical approaches since, except from basic examples, exact analytical solutions are rarely available. To achieve very accurate approximations, a variety of techniques have been utilised, including quadrature schemes, spectral methods, wavelet-based approaches, collocation methods, and Laplace transform methods. These systems are suitable for recursive numerical techniques because the Volterra structure permits step-by-step integration of time. On the other hand, memory terms may drive up computing costs and storage needs, therefore researchers are constantly looking for ways to make numerical systems faster and more efficient with memory.

Systems having delays, feedback, and hereditary effects can be represented using Volterra integro-differential systems, which have applications in control theory and systems engineering. Chemical reactors, biological control mechanisms, thermal systems, and other processes having aftereffects can be described using this system's natural framework in control applications. Research on the optimum control, observability, and controllability of Volterra systems has recently grown in popularity, helping to close the gap between theoretical mathematics and real-world engineering. Stochastic Volterra integro-differential systems, which capture uncertainty and random disturbances in physical processes, have also been studied as a result of stochastic elements' integration.

Machine learning, signal processing, and network dynamics are a few of the newer areas that have discovered uses for Volterra integro-differential systems. Because inputs from the past affect outputs from the present, memory-dependent models are finding more and more use in adaptive systems and neural networks. By including fractional derivatives and integrals, fractional-order Volterra integro-differential systems have enhanced the capacity for modelling anomalous diffusion and long-range temporal correlations. These advancements demonstrate how Volterra systems may be used to tackle modern scientific and technical problems.

II. REVIEW OF LITERATURE

Rani, Mamta & Manchanda, Pammy (2024) an approach to solving systems of Volterra integral equations using numerical methods based on Legendre wavelets has been introduced. The provided issue is reduced to a system of algebraic equations using the Legendre wavelets approximation and the Gauss integration method. Next, we use Newton's technique to solve this system of equations and discover the coefficients that are unknown. Several numerical examples are provided to demonstrate the suggested strategy. The results of a comparative error analysis are also available.

Sateesha, Channaveerapala & H., Manjula (2021) our approach to solving a system of Volterra integro-differential equations (SVIDE) is a Haar wavelet collocation technique, which we introduce in this study. The key feature of this method is that it simplifies these types of problems to systems of algebraic equations. The effectiveness and usefulness of the approach are tested using a small number of challenges. To prove the validity of the suggested approach, numerical results accompanied by comparisons are provided.

Wang, Yanxin & Zhu, Li. (2017) an algorithm for solving fractional-order nonlinear Volterra integro-differential equations using wavelets is introduced. It relies on approximations of the kind used by Euler wavelets. The first step is to introduce the Euler wavelet and then develop an operational matrix for fractional-order integration. It is possible to solve the nonlinear fractional integro-differential equations using existing numerical techniques by reducing them to a system of algebraic equations using the operational matrix. A number of solutions have also been investigated, some of which exhibit smooth behaviour, others non-smooth behaviour, and even unique behaviour. The technique's validity and applicability are illustrated with the use of examples.

Vanani, S. & Aminataei, Azim (2011) a numerical approach to solving integro-differential equations of Volterra type is laid forth. To solve the Volterra integro-differential equations, we first transform them into power series, and then we translate those power series into Padé series form. This provides us an arbitrary order. The next part introduces a method for efficiently estimating the numerical solution's inaccuracy. The suggested method's great accuracy and efficiency are shown by certain trials and comparisons with other approaches. The method's great benefits are demonstrated by solving nonlinear Volterra integro-differential equations using this approach and by computing the run time. Furthermore, the approach is convergent, as shown by numerical trials.

Shahsavaran, Ahmad. (2010) The Volterra and Fredholm integro differential equations are solved using the Haar wavelet method in this study. The primary issue is distilled to a set of linear algebraic equations for this reason. We analyse four test problems for which we know the precise answer and conduct a thorough error analysis.

III. NUMERICAL FORMULATION OF THE LINEAR VOLTERRA INTEGRO-DIFFERENTIAL SYSTEM

Presenting the linear Volterra integro-differential system We get Eqn. at p,q = 1,2,...&a = 0 by

$$\left.\begin{array}{l} u_1^{(m)}(x) = \int_0^x k_{11}(x,t)u_1(t)dt - \int_0^x k_{12}(x,t)u_2(t)dt = f_1(x) \\ u_2^{(m)}(x) = \int_0^x k_{21}(x,t)u_1(t)dt - \int_0^x k_{22}(x,t)u_2(t)dt = f_2(x) \end{array}\right\} \quad (1)$$

Where

$${}_{\lambda}u_1^{(r)} = c_r, u_2^{(r)} = d_r, 0 \le r \le (m-1)$$

m is the order of derivative, $m \geq 1$

$$\left.\begin{array}{l} u_1^{(m)}(x) = \sum_{i=1}^{2^{J+1}} a_i h_i(x) \text{ then} \\ u_1(x) = \sum_{i=1}^{2^{J+1}} a_i p_{mi}(x) + \sum_{r=0}^{m-1} \frac{1}{r!} c_i x^r, \\ u_2^{(m)}(x) = \sum_{i=1}^{2^{J+1}} b_i h_i(x) \text{ then} \\ u_2(x) = \sum_{i=1}^{2^{J+1}} b_i p_{mi}(x) + \sum_{r=0}^{m-1} \frac{1}{r!} d_i x^r, \end{array}\right\} \quad (2)$$

$$\left.\begin{array}{l} \sum_{i=1}^{2^{J+1}} a_i h_i(x) - \sum_{i=1}^{2^{J+1}} a_i D_{1i}(x) - \sum_{i=1}^{2^{J+1}} b_i D_{2i}(x) = F_1(x) \\ \sum_{i=1}^{2^{J+1}} b_i h_i(x) - \sum_{i=1}^{2^{J+1}} a_i D_{3i}(x) - \sum_{i=1}^{2^{J+1}} b_i D_{4i}(x) = F_2(x) \end{array}\right\} \quad (3)$$

Substitute Eqn. (2) in Eqn. (2) we get

Here,

$$F_1(x) = f_1(x) + \sum_{r=0}^{m-1} \frac{1}{r!} c_r \int_0^x k_{11}(x,t) t^r dt + \sum_{r=0}^{m-1} \frac{1}{r!} d_r \int_0^x k_{12}(x,t) t^r dt$$

$$F_2(x) = f_2(x) + \sum_{r=0}^{m-1} \frac{1}{r!} c_r \int_0^x k_{21}(x,t) t^r dt + \sum_{r=0}^{m-1} \frac{1}{r!} d_r \int_0^x k_{22}(x,t) t^r dt$$

$$\left.\begin{array}{l} D_{1i}(x) = \int_0^x k_{11}(x,t) p_{mi}(t) dt \\ D_{2i}(x) = \int_0^x k_{12}(x,t) p_{mi}(t) dt \\ D_{3i}(x) = \int_0^x k_{21}(x,t) p_{mi}(t) dt \\ D_{4i}(x) = \int_0^x k_{22}(x,t) p_{mi}(t) dt \end{array}\right\} \quad (4)$$

Descriteze the Eqn. (4) at collocation points

$x_1 = 0.5(\tilde{x}_{s-1} + \tilde{x}_s) l = 1,2,\dots,2^{J+1}$, where $\tilde{x}_s$ is the grid point given by $\tilde{x}_s = a + s\Delta x, s = 0,1,\dots,2^{J+1}$. We get linear system of equations are as follows:

$$\left.\begin{array}{l} \sum_{i=1}^{2^{J+1}} a_i [h_i(x_l) - D_{1i}(x_l)] - \sum_{i=1}^{2^{J+1}} b_i D_{2i}(x_l) = F_1(x_l) \\ \sum_{i=1}^{2^{J+1}} b_i [h_i(x_l) - D_{4i}(x_l)] - \sum_{i=1}^{2^{J+1}} a_i D_{3i}(x_l) = F_2(x_l) \end{array}\right\} \quad (5)$$

The Eqn. (5) can be written as in the matrix form as

$$a[H - D_1] - bD_2 = F_1$$

$$a[H - D_4] - aD_3 = F_2$$

IV. NUMERICAL RESULTS AND DISCUSSION

To verify the method's validity and precision, we ran it on certain sample issues in this section. We compared the outcomes to those of previously published numerical approaches. We calculated the best approximations, the most precise answers, and the most possible absolute errors.

$$e_{J_1} = \left| u_1(x_l)_{\text{app}} - u_1(x_l)_{\text{exact}} \right|,$$

$$e_{J_2} = \left| u_2(x_l)_{\text{app}} - u_2(x_l)_{\text{exact}} \right|$$

Example: The Volterra integrodifferential equation system on the interval [0,1] is taken into account here.

$$\left.\begin{aligned} u_1^1(x) &= 1 - x^2 + e^x + \int_0^x (u_1(t) + u_2(t))\,dt \\ u_2^1(x) &= 3 - 3e^x + \int_0^x (u_1(t) + u_2(t))\,dt \end{aligned}\right\} \quad (6)$$

With the initial conditions $u_1(0) = 1$, $u_2(0) = -1$, its analytical solution is $u_1(x) = x+e^x$, $u_2(x) = x-e^x$

Table 1: Results of $u_1(x)$

x	$u_1(x)$Appr	$u_1$exact (x)	Absolute error by HWCM
0.1	1.2052	1.205170	3.0×10^{-5}
0.2	1.4214	1.421403	3.0×10^{-6}
0.3	1.6499	1.649859	4.1×10^{-5}
0.4	1.8918	1.891824	2.4×10^{-5}
0.5	2.1487	2.148721	2.1×10^{-5}
0.6	2.4221	2.422119	1.9×10^{-5}
0.7	2.7138	2.713753	4.7×10^{-5}
0.8	3.0255	3.025541	4.1×10^{-5}
0.9	3.3596	3.359603	3.0×10^{-6}

Table 2: Results of $u_2(x)$

x	$u_2(x)$Appr	u_2 exact (x)	Absolute error by HWCM
0.1	−1.0052	−1.005170	3.0×10^{-5}
0.2	−1.2214	−1.221403	3.0×10^{-6}
0.3	−1.3499	−1.349859	4.1×10^{-5}
0.4	−1.4918	−1.491824	2.4×10^{-5}
0.5	−1.6487	−1.648721	2.1×10^{-5}
0.6	−1.8221	−1.822119	1.9×10^{-5}
0.7	−2.0138	−2.013753	4.7×10^{-5}
0.8	−2.2255	−2.225541	4.1×10^{-5}
0.9	−2.4596	−2.459603	3.0×10^{-6}

V. CONCLUSION

In order to simplify the computational effort, the suggested technique uses appropriate collocation points to transform the original integro-differential system into a system of linear algebraic equations. A number of numerical examples show that the method's approximations accord very well with the precise analytical answers. The method's great precision and stability are demonstrated by the extremely modest maximum absolute errors. The results show that the wavelet approach is simple to use, produces accurate results with little computing load, and is computationally efficient. The suggested method can be generalized to solve higher-order, nonlinear, multi-dimensional Volterra integro-differential systems because of its accuracy and flexibility. For many scientific and engineering issues, the wavelet-based method is an effective numerical tool.

REFERENCES

1. M. Rani and P. Manchanda, "Legendre wavelet method for systems of Volterra integral equations," *Library Progress (International)*, vol. 44, no. 2, pp. 2–14, 2024.
2. Ali, M. Yaseen, and S. Khan, "Addressing Volterra partial integro-differential equations through an innovative extended cubic B-spline collocation technique," *Symmetry*, vol. 1, no. 1, pp.1851, 2023.
3. G. Ajileye and S. Amoo, "Numerical solution to Volterra integro-differential equations using collocation approximation," vol. 9, no. 5, pp. 1–8, 2023.
4. N. A. Mohammad, Y. A. Sabawi, and M. S. Hasso, "Haar wavelet method for the numerical solution of non-linear Fredholm integro-differential equations," *Journal of Education and Science*, vol. 32, no. 4, pp. 10–25, 2023.
5. Kumar, K. Deswal, and S. Singh, "Wavelet-based approximation with nonstandard finite difference scheme for singularly perturbed partial integro-differential equations," *Computational and Applied Mathematics*, vol. 41, no. 8, pp. 341, 2022.
6. Hussain, "Existence and numerical analysis using Haar wavelet for fourth-order multi-term fractional differential equations," *Computational and Applied Mathematics*, vol. 41, no. 7, pp. 32-9, 2022.
7. Sateesha and H. Manjula, "Numerical approach for the solution of system of Volterra integro-differential equations using wavelet method," *Malaya Journal of Matematik*, vol. S, no. 1, pp. 425–429, 2021.
8. S. Saleem, I. Aziz, and M. Z. Hussain, "A simple algorithm for numerical solution of nonlinear parabolic partial differential equations," *Engineering with Computers*, vol. 36, no. 4, pp. 1763–1775, 2020.
9. M. Ahsan, I. Ahmad, M. Ahmad, and I. Hussain, "A numerical Haar wavelet–finite difference hybrid method for linear and non-linear Schrödinger equation," *Mathematics and Computers in Simulation*, vol. 165, no. 4, pp. 13–25, 2019.
10. F. Mirzaee and S. Alipour, "Numerical solution of non-linear partial quadratic integro-differential equations of fractional order via hybrid of block-pulse and parabolic functions," *Numerical Methods for Partial Differential Equations*, vol. 35, no. 3, pp. 1134–1151, 2019.
11. Aziz and I. Khan, "Numerical solution of diffusion and reaction–diffusion partial integro-differential equations," *International Journal of Computational Methods*, vol. 15, no. 6, pp. 1850047, 2018.
12. Y. Wang and L. Zhu, "Solving nonlinear Volterra integro-differential equations of fractional order by using Euler wavelet method," *Advances in Difference Equations*, vol. 20, no. 1, pp. 2–15, 2017.
13. R. Loh and C. Phang, "A new numerical scheme for solving system of Volterra integro-differential equation," *Alexandria Engineering Journal*, vol. 57, no. 6, pp. 47-50, 2016,
14. S. S. Ray and P. Sahu, "A numerical approach for solving nonlinear fractional Volterra–Fredholm integro-differential equations with mixed boundary conditions," *International Journal of Wavelets, Multiresolution and Information Processing*, vol. 14, no. 6, pp. 7-10, 2016.
15. R. Chandel, A. Singh, and D. Chouhan, "Solution of higher order Volterra integro-differential equations by Legendre wavelets," *International Journal of Applied Mathematics*, vol. 28, no. 5, pp. 16-20, 2015.
16. G. B., "A numerical solution of nonlinear parabolic-type Volterra partial integro-differential equations using radial basis functions," *Engineering Analysis with Boundary Elements*, vol. 36, no. 5, pp. 881–893, 2012.
17. S. Vanani and A. Aminataei, "Numerical solution of Volterra integro-differential equations," *Journal of Computational Analysis and Applications*, vol. 13, no. 1, pp. 2–15, 2011.
18. A. Shahsavaran, "Numerical solution of linear Volterra and Fredholm integro-differential equations using Haar wavelets," vol. 6, no. 1, pp. 85–96, 2010.